Remark: Each question carries 25 marks. In questions with two parts, the parts
carry equal marks.

1. State and prove Desargues's Theorem in $PG(2, q), q$ prime power.

2. A one - factor in a graph is a spanning sub graph which is regular of degree 1. A
   one - factorization is a partition of the edge set into one - factors. Consider the
   incidence system whose points are the vertices and one - factors of K_6, whose
   blocks are the edges and one - factorizations of K_6, and incidence is belongs to
   or its reverse, whichever makes sense. Show that this is a projective plane.

3. Let $f(G)$ denote the Perron eigenvalue of a connected graph G.

   (a) Show that $f(H) < f(G)$ for all proper connected subgraphs H of G.

   (b) Classify all connected graphs G with $f(G) = 2$.

4. (a) Let $0 \leq t < k < v$ and $\lambda > 0$ be integers. A $t - (v, k, \lambda)$ design is an
       incidence system with v points and blocks of size k each such that each
       t-set of points is contained in exactly λ blocks. Show that for $0 \leq s \leq t$, a
       $t - (v, k, \lambda)$ design is also an $s - (v, k, \lambda_s)$ design, where λ_s is a number to
       be computed.

   (b) Let $D = (V, \mathbb{B})$ be a $2 - (4n - 1, 2n - 1, n - 1)$ design, $n \geq 2$. Take a
       symbol $*$ not in V. Put $\bar{V} = V \sqcup \{*\}$ and $\bar{\mathbb{B}} = \mathbb{B}_1 \sqcup \mathbb{B}_2$ where

       $$\mathbb{B}_1 = \{B \sqcup \{*\} : B \in \mathbb{B}\}$$

       $$\mathbb{B}_2 = \{V \backslash B : B \in \mathbb{B}\}.$$

       Show that $\bar{D} = (\bar{V}, \bar{\mathbb{B}})$ is a 3 - design. Compute its parameters.